数理解析研究所講究録
第 1547 巻 2007 年 114-119

Positive solutions for semilinear elliptic equations involving Dirac measures

神戸大学・工学部 内藤 雄基 (Yūki Naito)
Faculty of Engineering, Kobe University

東北大学大学院・理学研究科 佐藤 得志 (Tokushi Sato)
Mathematical Institute, Tohoku University

We are concerned with the problem of finding positive solutions with prescribed isolated singularities to semilinear elliptic equations. Choosing a finite set of points $\{a_i\}_{i=1}^m$ in $\mathbf{R}^N$ and a set of positive numbers $\{c_i\}_{i=1}^m$, we consider the existence of positive solutions of the problem

$$-\Delta u+u=u^p+\kappa\sum_{i=1}^m c_i\delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N), \tag{1.1$_\kappa$}$$

with the condition at infinity

$$u(x)\to 0 \quad \text{as } |x|\to\infty, \tag{1.2}$$

where $N\geq 3$, $1<p<N/(N-2)$, $\kappa\geq 0$ is a parameter, and δ_a is the Dirac delta function supported at $a\in\mathbf{R}^N$. We denote the Laplacian on $\mathbf{R}^N$ by Δ and the class of distributions on $\mathbf{R}^N$ by $\mathcal{D}'(\mathbf{R}^N)$.

We recall some known results concerning the singularities of possible solutions of the equation. Let Ω be a bounded domain in $\mathbf{R}^N$ containing 0. By the works due to Lions [14] and Brezis and Lions [6], we obtain the following result.

Theorem A [14, 6]. *Assume that $u\in C^2(\Omega\setminus\{0\})$ satisfies*

$$-\Delta u+u=u^q \quad \text{in } \Omega\setminus\{0\} \tag{1.3}$$

with $q>1$ and $u\geq 0$ a.e. in Ω. Then $u\in L^q_{\mathrm{loc}}(\Omega)$ and

$$-\Delta u+u=u^q+\kappa\delta_0 \quad \text{in } \mathcal{D}'(\Omega) \tag{1.4}$$

for some $\kappa\geq 0$. Furthermore, the following (i) *and* (ii) *hold.*

(i) *In the case $1<q<N/(N-2)$, if $\kappa=0$ in (1.4) then $u\in C^2(\Omega)$, and if $\kappa>0$ then u behaves like a multiple of the fundamental solution E_0 for $-\Delta$ in $\mathbf{R}^N$, i.e., $-\Delta E_0=\delta_0$ in $\mathcal{D}'(\mathbf{R}^N)$.*

(ii) *In the case $q \geq N/(N-2)$, there holds $\kappa = 0$ in* (1.4).

For the proof, see Theorem 1 in [6] and Corollary 1, Theorem 2, and Remark 2 in [14].

It should be mentioned that Johnson, Pan, and Yi [13] showed the existence and asymptotic behaviour of singular positive radial solution u of (1.3) with $1 < q < (N+2)/(N-2)$. In particular, they showed that, if $N/(N-2) < q < (N+2)/(N-2)$, there exists a positive solution u of (1.3) satisfying $u(x) \sim c|x|^{-2/(p-1)}$ as $|x| \to 0$ for some constant $c > 0$. Then, in this case, the singularity of u at $x = 0$ exists, but is not visible in the sense of distribution.

In this paper, we investigate the existence of positive solutions with prescribed isolated singularities to the equation in $\mathbf{R}^N$. By (ii) of Theorem A, if $p \geq N/(N-2)$ then $(1.1)_\kappa$ with $\kappa > 0$ has no positive solution $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. Hence, the condition $1 < p < N/(N-2)$ is necessary for the existence of positive solutions $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ of $(1.1)_\kappa$ with $\kappa > 0$.

We review some known results concerning related problems. Lions [14] studied the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = u^p + \kappa\delta_0 & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbf{R}^N$ containing 0 with smooth boundary $\partial\Omega$. It was shown in [14] that there exists $\kappa^* > 0$ such that (1.5) has at least two positive solutions for each $\kappa \in (0, \kappa^*)$ and no such solution for $\kappa > \kappa^*$. Later, Baras and Pierre [4] studied the existence of positive solutions for the problem

$$\begin{cases} -\Delta u = u^p + \kappa\mu & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where μ is a positive bounded Radon measure in Ω. In [4] they showed that (1.6) has at least one positive solution for each sufficiently small $\kappa > 0$ by investigating the corresponding integral equations. See also Roppongi [16]. Amann and Quittner [3] exhibited the existence of $\kappa^* > 0$ such that (1.6) has at least two positive solutions for $0 < \kappa < \kappa^*$ and no solution for $\kappa > \kappa^*$. Bidaut-Veron and Yarur [5] gave the existence results and a priori estimates for (1.6) including the case where μ is unbounded. In [3], [5], they also consider the problems involving measures as boundary data. We also refer a survey by Veron [19], [20], and the references therein. In [17] the second author studied the existence of positive solutions for the problem

$$-\Delta u + f(u) = \sum_{i=1}^{m} c_i\delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N)$$

in the cases where f is nonnegative. In [17] he also showed the nonexistence of positive solutions for some f with sign changing.

Concerning nonhomogeneous semilinear elliptic problems of the form

$$-\Delta u + u = u^q + \kappa f(x) \quad \text{in } \mathbf{R}^N$$

with $q > 1$ and $f \in H^{-1}(\mathbf{R}^N)$, we refer to Zhu [21], Deng and Li [10], [11], Cao and Zhou [7], and Hirano [12]. They successfully showed the existence of at least two positive solutions of the problems under suitable conditions. See also [18, 8] for closely related problems.

In order to state our results, we introduce some notations. Let E_1 denote the fundamental solution for $-\Delta + I$ in $\mathbf{R}^N$, that is,

$$E_1(x) = E_1(|x|) = \frac{1}{(2\pi)^{N/2}|x|^{(N-2)/2}} K_{(N-2)/2}(|x|) \quad \text{for } x \in \mathbf{R}^N \setminus \{0\},$$

where K_ν is the modified Bessel function of order ν. We see that E_1 has the following properties:

$$E_1(x) \sim \frac{1}{(N-2)N\omega_N|x|^{N-2}} \quad \text{as } |x| \to 0, \quad \text{and}$$
$$E_1(x) \sim c_1|x|^{-(N-1)/2}e^{-|x|} \quad \text{as } |x| \to \infty,$$

where ω_N denotes the volume of the unit ball in $\mathbf{R}^N$ and $c_1 > 0$ is a constant depends on N. In particular, $E_1 \in C^\infty(\mathbf{R}^N \setminus \{0\})$ and $E_1 \in L^r(\mathbf{R}^N)$ for all $1 \le r < N/(N-2)$. Define f_0 by

$$f_0(x) = \sum_{i=1}^m c_i E_1(x - a_i).$$

Then $f_0 \in C^\infty(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $f_0 \in L^r(\mathbf{R}^N)$ for all $1 \le r < N/(N-2)$, and f_0 satisfies

$$-\Delta f_0 + f_0 = \sum_{i=1}^m c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

In this paper we refer to u as a positive solution of $(1.1)_\kappa$ if $u \in L^p_{\text{loc}}(\mathbf{R}^N)$ satisfies $(1.1)_\kappa$ in the sense of distribution and $u > 0$ a.e. in $\mathbf{R}^N$.

Proposition 1.1. *Let $u \in L^p_{\text{loc}}(\mathbf{R}^N)$ be a positive solution of $(1.1)_\kappa$ with $\kappa > 0$. Then $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $u(x) > 0$ for $x \in \mathbf{R}^N \setminus \{a_i\}_{i=1}^m$. Assume, in addition, that (1.2) holds. Then $u \in L^q(\mathbf{R}^N)$ for all $q \in [1, N/(N-2))$ and u satisfies*

$$u = E_1 * [u^p] + \kappa f_0 \quad \text{a.e. in } \mathbf{R}^N \tag{1.7$_\kappa$}$$

and $u(x) = O(E_1(x))$ as $|x| \to \infty$, where the symbol $*$ denotes the convolution.

For each $\kappa > 0$, we define U_j^κ for $j = 0, 1, 2, \ldots$, inductively, by

$$U_0^\kappa = \kappa f_0 \quad \text{and} \quad U_j^\kappa = E_1 * \left[(U_{j-1}^\kappa)^p\right] + \kappa f_0 \quad \text{for } j = 1, 2, \ldots. \tag{1.8}$$

Take $q_0 \in (p, N/(N-2))$ arbitrarily, and define $\{q_j\}$ by

$$\frac{1}{q_j} = \frac{1}{q_0} - \left(\frac{2}{N} - \frac{p-1}{q_0}\right) j = \frac{1}{q_{j-1}} - \left(\frac{2}{N} - \frac{p-1}{q_0}\right) \quad \text{for } j = 1, 2, \ldots. \tag{1.9}$$

From $p < N/(N-2)$ and $q_0 > p$, it follows that $2/N - (p-1)/q_0 > 0$. Then, by choosing suitable q_0 if necessary, there exists an positive integer denoted by j_0 satisfying

$$\frac{1}{q_{j_0-1}} > 0 > \frac{1}{q_{j_0}}. \tag{1.10}$$

We use the notation $C_0(\mathbf{R}^N) = \{u \in C(\mathbf{R}^N) : u(x) \to 0 \quad \text{as } |x| \to \infty\}$.

Proposition 1.2. *For each $\kappa \in (0, \infty)$, the following* (i) -(iii) *are equivalent to each other* :

(i) $u = w + U^{\kappa}_{j_0} \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ *is a positive solution of* $(1.1)_\kappa$–(1.2);

(ii) $w \in C_0(\mathbf{R}^N)$ *is positive in* $\mathbf{R}^N$ *and satisfies*

$$w = E_1 * \left[(w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p\right] \quad \textit{in } \mathbf{R}^N; \tag{1.11$_\kappa$}$$

(iii) $w \in H^1(\mathbf{R}^N)$ *is a weak positive solution of*

$$-\Delta w + w = (w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p \quad \textit{in } \mathbf{R}^N, \tag{1.12$_\kappa$}$$

that is, $w > 0$ a.e. in $\mathbf{R}^N$ and satisfies

$$\int_{\mathbf{R}^N} (\nabla w \cdot \nabla \psi + w\psi)\, dx = \int_{\mathbf{R}^N} \left((w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p\right) \psi dx \tag{1.13$_\kappa$}$$

for any $\psi \in H^1(\mathbf{R}^N)$.

By Proposition 1.2, the problem $(1.1)_\kappa$–(1.2) can be reduced to the problems $(1.11)_\kappa$ in $C_0(\mathbf{R}^N)$ and $(1.12)_\kappa$ in $H^1(\mathbf{R}^N)$. We will investigate the problems $(1.11)_\kappa$ and $(1.12)_\kappa$ by an approach based on adaptation of the methods by [1, 2, 9, 14].

Our main results are stated in the following theorems.

Theorem 1. *There exists $\kappa^* \in (0, \infty)$ such that*

(i) *if $0 < \kappa < \kappa^*$ then the problem* $(1.1)_\kappa$–(1.2) *has a positive minimal solution $\underline{u}_\kappa$, that is, $\underline{u}_\kappa \le u$ a.e. in $\mathbf{R}^N$ for any positive solution u of* $(1.1)_\kappa$–(1.2). *Furthermore, if $0 < \kappa < \hat{\kappa} < \kappa^*$ then $\underline{u}_\kappa < \underline{u}_{\hat{\kappa}}$ a.e. in $\mathbf{R}^N$;*

(ii) *if $\kappa > \kappa^*$ then the problem* $(1.1)_\kappa$–(1.2) *has no positive solution.*

Theorem 2. *If $\kappa = \kappa^*$ then the problem* $(1.1)_\kappa$–(1.2) *has a unique positive solution.*

Theorem 3. *If $0 < \kappa < \kappa^*$ then the problem* $(1.1)_\kappa$–(1.2) *has a positive solution $\overline{u}_\kappa$ satisfying $\overline{u}_\kappa > \underline{u}_\kappa$.*

Proofs of Theorems 1-3 can be found in [15]. In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of $(1.12)_\kappa$ and $(1.11)_\kappa$, respectively, to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of $(1.12)_\kappa$. We will prove Theorem 3 by employing the variational method with the Mountain Pass Lemma. In the proofs of Theorems 2 and 3, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions play a crucial role.

REFERENCES

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. **18** (1976) 620–709.

[2] H. Amann, Supersolutions, monotone iterations, and stability, J. Differential Equations **21** (1976) 363–377.

[3] H. Amann and P. Quittner, Elliptic boundary value problems involving measures: existence, regularity, and multiplicity, Adv. Differential Equations **3** (1998) 753–813.

[4] P. Baras and M. Pierre, Critere d'existence de solutions positives pour des equations semi-lineaires non monotones, Ann. Inst. H. Poincare Anal. Non Lineaire **2** (1985) 185–212.

[5] M. F. Bidaut-Veron and C. Yarur, Semilinear elliptic equations and systems with measure data: existence and a priori estimates, Adv. Differential Equations **7** (2002) 257–296.

[6] H. Brezis and P.-L. Lions, A note on isolated singularities for linear elliptic equations, Mathematical analysis and applications, Part A, pp. 263–266, Adv. in Math. Suppl. Stud., **7a**, Academic Press, New York-London, 1981.

[7] D-M. Cao and H.-S. Zhou, Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996) 443–463.

[8] D-M. Cao and H.-S. Zhou, On the existence of multiple solutions of nonhomogeneous elliptic equations involving critical Sobolev exponents, Z. Angew. Math. Phys. **47** (1996) 89–96.

[9] M. G. Crandall and P. H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Rational Mech. Anal. **58** (1975) 207–218.

[10] Y. Deng and Y. Li, Existence and bifurcation of the positive solutions for a semilinear equation with critical exponent, J. Differential Equations **130** (1996) 179–200.

[11] Y. Deng and Y. Li, Existence of multiple positive solutions for a semilinear elliptic equation, Adv. Differential Equations **2** (1997) 361–382.

[12] N. Hirano, Existence of entire positive solutions for nonhomogeneous elliptic equations, Nonlinear Anal. **29** (1997) 889–901.

[13] R. A. Johnson, X. B. Pan, and Y. Yi, Singular solutions of the elliptic equation $\Delta u - u + u^p = 0$, Ann. Mat. Pura Appl. **166** (1994) 203–225.

[14] P.-L. Lions, Isolated singularities in semilinear problems, J. Differential Equations **38** (1980) 441–450.

[15] Y. Naito and T. Sato, Positive solutions for semilinear elliptic equations with singular forcing terms, J. Differential Equations (to appear)

[16] S. Roppongi, On singular solutions for a semilinear elliptic equation, Kodai Math. J. **18** (1995) 44–56.

[17] T. Sato, Positive solutions with weak isolated singularities to some semilinear elliptic equations, Tohoku Math. J. **47** (1995) 55–80.

[18] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincare Anal. Non Lineaire **9** (1992) 281–304.

[19] L. Veron, Singularities of solutions of second order quasilinear equations, Pitman Research Notes in Mathematics Series, **353** Longman, Harlow, 1996.

[20] L. Veron, Elliptic equations involving measures, Stationary partial differential equations. I, 593–712, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[21] X. P. Zhu, A perturbation result on positive entire solutions of a semilinear elliptic equation, J. Differential Equations **92** (1991) 163–178.